Spectral analysis of a flat plasma sheet model

2005 J. Phys. A: Math. Gen. 38 11027

(http://iopscience.iop.org/0305-4470/38/50/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 11027–11043 doi:10.1088/0305-4470/38/50/011

Spectral analysis of a flat plasma sheet model

M Bordag[1], I G Pirozhenko[2] and V V Nesterenko[2]

[1] Institute for Theoretical Physics, University of Leipzig, Augustusplatz 10/11, 04109 Leipzig, Germany
[2] Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia

E-mail: Michael.Bordag@itp.uni-leipzig.de, pirozhen@thsun1.jinr.ru and nestr@thsun1.jinr.ru

Received 24 August 2005, in final form 21 October 2005
Published 30 November 2005
Online at stacks.iop.org/JPhysA/38/11027

Abstract
The spectral analysis of the electromagnetic field on the background of an infinitely thin flat plasma layer is carried out. This model loosely imitates a single base plane from graphite and it is of interest for theoretical studies of fullerenes. By making use of the Hertz potentials the solutions to Maxwell equations with the appropriate matching conditions at the plasma layer are derived and on this basis the spectrum of electromagnetic oscillations is determined. The model is naturally split into the TE-sector and TM-sector. Both the sectors have positive continuous spectra, but the TM-modes have in addition a bound state, namely, the surface plasmon. This analysis relies on the consideration of the scattering problem in the TE- and TM-sectors. The spectral zeta-function and integrated heat kernel are constructed for different branches of the spectrum in an explicit form. As a preliminary, the rigorous procedure of integration over the continuous spectra is formulated by introducing the spectral density in terms of the scattering phase shifts. The asymptotic expansion of the integrated heat kernel at small values of the evolution parameter is derived. By making use of the technique of integral equations, developed earlier by the same authors, the local heat kernel (Green's function or fundamental solution) is also constructed. As a by-product, a new method is demonstrated for deriving the fundamental solution to the heat conduction equation (or to the Schrödinger equation) on an infinite line with the δ-like source. In particular, for the heat conduction equation on an infinite line with the δ-source a nontrivial counterpart is found, namely, a spectral problem with point interaction, which possesses the same integrated heat kernel while the local heat kernels (fundamental solutions) in these spectral problems are different.

PACS numbers: 11.10.Gh, 42.59.Pq, 04.62.+v, 11.10.−z, 02.40.−k

1. Introduction

The use of fullerenes in practical applications is increasingly important (in superconducting techniques, in mechanical engineering for reducing friction, in medicine for drug production and so on). Fullerenes [1, 2] are the third crystal modification of carbon. They were first produced in the 1980s (Nobel Prize in Chemistry in 1996, R F Curl, H W Kroto and R E Smalley). The elementary blocks of the fullerenes are the giant carbon molecules (C_{60}, C_{70}, C_{84} and larger) having the form of empty balls, ellipsoids, tubes and so on. There is a special branch of technology, nanotechnology [3], which is based on using the fullerenes, and especially nanotubes. In theoretical studies of the fullerenes it is important to estimate the ground state energy (the Casimir energy) of such molecules. The carbon shells, forming the structure of fullerenes, can be considered as infinitely thin layers, and their collective electrons can be treated as a two-dimensional plasma layer by making use of the hydrodynamic picture for its description. As a result, the problem reduces to the solution of the Maxwell equations with charges and currents distributed along the surfaces modelling the carbon sheets. The method of boundary (matching) conditions applies here, i.e., the Maxwell equations are considered only outside the plasma layers and when approaching these sheets the electric and magnetic fields should meet appropriate matching conditions. From the mathematical standpoint the pertinent spectral problem proves to be well posed.

For constructing the quantum version of this model and, first of all, for a rigorous treatment of the divergences encountered here the spectral analysis of the underlying field theory should be accomplished; i.e., the spectrum of the dynamical system in question should be determined, the mathematically consistent integration procedure over this spectrum should be defined and on this basis the spectral zeta-function and heat kernels (integrated and local ones) should be calculated as well as their asymptotic expansions under small values of the evolution parameter should be found.

The present work is devoted to the spectral analysis of an infinitely thin plasma sheet model with the simplest geometry; namely, the plasma layer has the form of a plane. The spectrum of the model proves to be very rich; namely, it contains both continuous branches and bound states (surface plasmon), and we managed to construct all the pertinent spectral functions possessing interesting properties.

The outline of the paper is as follows. In section 2 the physical formulation of the model under study is briefly given. In section 3 the solutions to the Maxwell equations obeying the pertinent matching conditions at the plasma layer are constructed. The use of the Hertz vectors for this aim proves to be very effective. Proceeding from this the spectrum of electromagnetic oscillations in the model is found. As usual in electrodynamical problems the model is naturally separated into the TE-sector and TM-sector. Both the sectors have positive continuous spectra, but the TM-modes have in addition a bound state, namely, the surface plasmon. This analysis relies on the consideration of the scattering problem in the TE- and TM-sectors. In section 4 the spectral zeta-function and integrated heat kernel are constructed for different branches of the spectrum in an explicit form. As a preliminary, the rigorous procedure of integration over the continuous spectra is formulated by introducing the spectral density (the function of the spectral shift) in terms of the scattering phase shifts. Section 5 is devoted to the construction of the local heat kernel (Green's function or fundamental solution) in the model at hand. Here the technique of integral equations governing the heat kernel to be found is used. As a by-product, a new method is demonstrated here for deriving the fundamental solution to the heat conduction equation (or to the Schrödinger equation) on an infinite line with the δ-like source. In particular, for the heat conduction equation on an infinite line with the δ-source a nontrivial counterpart is found, namely, a spectral problem with point interaction, which possesses the same integrated heat kernel. However, the local heat kernels in these spectral problems are different. In the conclusion (section 6) the obtained results are summarized briefly.

2. Formulation of the model

In recent papers [4, 5] Barton has proposed and investigated the model of an infinitesimally thin two-dimensional plasma layer that is loosely inspired by considering a single base plane from graphite or the giant carbon molecule C_{60}. Effectively, the model is described by the Maxwell equations with charges and currents distributed along the surface Σ:

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} - \mathrm{i}\omega \mathbf{B}/c = 0, \tag{2.1}$$

$$\nabla \cdot \mathbf{E} = 4\pi\delta(\mathbf{x} - \mathbf{x}_\Sigma)\sigma, \qquad \nabla \times \mathbf{B} + \mathrm{i}\omega\mathbf{E}/c = 4\pi\delta(\mathbf{x} - \mathbf{x}_\Sigma)\mathbf{J}/c. \tag{2.2}$$

It is assumed that the time variation of all the dynamical variables is described by a common factor $\mathrm{e}^{-\mathrm{i}\omega t}$.

The properties of the plasma layer (for example, plasma oscillations and screening) differ considerably from those in bulk media [6]. In hydrodynamic approach the plasma is considered as the electron fluid embedded in a rigid uniform positive background. Electrical neutrality requires that the equilibrium electron charge density $-en_0$ precisely cancels that of the background. In a dynamical situation, the electron density is altered to $n(\mathbf{x}, t)$, and the equation of continuity requires

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\dot{\boldsymbol{\xi}}) = 0, \tag{2.3}$$

where $\dot{\boldsymbol{\xi}}(\mathbf{x}, t)$ is the electron velocity at the point $\mathbf{x}$. Usually, one considers the *linear* response of an initially stationary system to an applied perturbation. In this case, the induced charge density

$$\sigma(\mathbf{x}, t) \equiv -e(n(\mathbf{x}, t) - n_0), \tag{2.4}$$

the velocity $\dot{\boldsymbol{\xi}}$, and the fields $\mathbf{E}$ and $\mathbf{B}$ are of first order. In this approximation, the continuity equation (2.3) gives for the plasma layer

$$\dot{\sigma} - en_0\nabla_\parallel \cdot \dot{\boldsymbol{\xi}} = 0 \qquad \text{or} \qquad \sigma = en_0\nabla_\parallel \cdot \boldsymbol{\xi}. \tag{2.5}$$

The superscripts $\parallel$ and $\perp$ indicate here and below the vector components respectively parallel and normal to the surface Σ.

For the induced charge current $\mathbf{J}$ we have

$$\mathbf{J} = -en_0\dot{\boldsymbol{\xi}} = \mathrm{i}e\omega n_0\boldsymbol{\xi}. \tag{2.6}$$

Newton's second law applied to an individual electron gives

$$m\ddot{\boldsymbol{\xi}}(\mathbf{x}, t) = -e\,\mathbf{E}_\parallel(\mathbf{x}, t), \qquad \mathbf{x} \in \Sigma, \qquad \text{or} \quad \boldsymbol{\xi} = \frac{e}{m\omega^2}\mathbf{E}_\parallel. \tag{2.7}$$

Finally, the induced-on-the-surface charges and electric currents are determined by the parallel components of electric field

$$\sigma = \frac{e^2 n_0}{m\omega^2}\nabla_\parallel \cdot \mathbf{E}_\parallel, \qquad \mathbf{J} = \mathrm{i}\frac{e^2 n_0}{m\omega}\mathbf{E}_\parallel. \tag{2.8}$$

As is known [7], electromagnetic field, generated by charges and currents with singular densities is described mathematically in the following way: outside the singularities the Maxwell equations (2.1), (2.2) without sources should be satisfied,

$$\nabla\cdot\mathbf{B}=0,\qquad \nabla\times\mathbf{E}-\mathrm{i}\omega\mathbf{B}/c=0, \tag{2.9}$$

$$\nabla\cdot\mathbf{E}=0,\qquad \nabla\times\mathbf{B}+\mathrm{i}\omega\mathbf{E}/c=0,\qquad \mathbf{x}\notin\Sigma, \tag{2.10}$$

and when approaching at singularities the limiting values of the fields should meet the matching conditions determined by the singular sources. In the regions of singularities the Maxwell equations are not considered. The singular charges and currents on the right-hand side of the Maxwell equations (2.2) lead to the following matching conditions:

$$[\mathbf{E}_{\parallel}]=0,\qquad [\mathbf{E}_{\perp}]=2q(c/\omega)^2\nabla_{\parallel}\cdot\mathbf{E}_{\parallel}, \tag{2.11}$$

$$[\mathbf{B}_{\perp}]=0,\qquad [\mathbf{B}_{\parallel}]=-2\mathrm{i}q(c/\omega)\mathbf{n}\times\mathbf{E}_{\parallel}. \tag{2.12}$$

Here q is a characteristic wavenumber $q=2\pi ne^2/mc^2$, the square brackets $[\mathbf{F}]$ denote the discontinuity of the field $\mathbf{F}$ when crossing the surface Σ, and $\mathbf{n}$ is a unit normal to this surface usually used in formulation of matching conditions [7].

The physical origin of the model leads to a Debye-type cutoff K on the surface-parallel wavenumbers of waves that the fluid can support. However, if one digresses from this cutoff then the Maxwell equations (2.9) and (2.10) with matching conditions (2.11) and (2.12) can be regarded as a local quantum field model that is interesting in itself. First of all, the question arises here concerns the analysis of divergences. As is known [8–13], to this end one has to calculate the heat kernel coefficients for the differential operator garnering the dynamics in this model (see equations (2.9)–(2.12)).

3. Solution to the Maxwell equations for a flat plasma sheet

When constructing the normal modes in this problem it is convenient to use the Hertz potentials [7, 14, 15]. In view of high symmetry of the flat plasma layer the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are expressed in terms of electric ($\mathbf{\Pi}'$) and magnetic ($\mathbf{\Pi}''$) Hertz vectors possessing only one nonzero component

$$\mathbf{\Pi}'=\mathbf{e}_z\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Phi(z),\qquad \mathbf{\Pi}''=\mathbf{e}_z\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Psi(z). \tag{3.1}$$

Here the plasma layer is taken as the (x, y) coordinate plane, the z-axis is normal to this plane, $\mathbf{k}$ is a two-component wave vector parallel to the plasma sheet and $\mathbf{s}=(x, y)$; $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are unit base vectors in this coordinate system. The common time-dependent factor $\mathrm{e}^{-\mathrm{i}\omega t}$ is dropped.

For given Hertz vectors $\mathbf{\Pi}'$ and $\mathbf{\Pi}''$ the electric and magnetic fields are constructed by the formulae

$$\mathbf{E}=\nabla\times\nabla\times\mathbf{\Pi}',\qquad \mathbf{B}=-\mathrm{i}\frac{\omega}{c}\nabla\times\mathbf{\Pi}'\quad(\text{TM-modes},\ B_z=0), \tag{3.2}$$

$$\mathbf{E}=\mathrm{i}\frac{\omega}{c}\nabla\times\mathbf{\Pi}'',\qquad \mathbf{B}=\nabla\times\nabla\times\mathbf{\Pi}''\quad(\text{TE-modes},\ E_z=0). \tag{3.3}$$

Substituting here $\mathbf{\Pi}'$ and $\mathbf{\Pi}''$ from equation (3.1) one obtains for the TE-modes

$$\mathbf{E}=(-k_y\mathbf{e}_x+k_x\mathbf{e}_y)\frac{\omega}{c}\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Psi(z), \tag{3.4}$$

$$\mathbf{B}=\mathrm{i}(k_x\mathbf{e}_x+k_y\mathbf{e}_y)\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Psi'(z)+\mathbf{e}_zk^2\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Psi(z) \tag{3.5}$$

and for the TM-modes

$$\mathbf{E}=\mathrm{i}(k_x\mathbf{e}_x+k_y\mathbf{e}_y)\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Phi'(z)+\mathbf{e}_zk^2\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Phi(z) \tag{3.6}$$

$$\mathbf{B}=(k_y\mathbf{e}_x-k_x\mathbf{e}_y)\frac{\omega}{c}\,\mathrm{e}^{\mathrm{i}\mathbf{ks}}\Phi(z). \tag{3.7}$$

The fields (3.4)–(3.7) will obey the Maxwell equations (2.9) and (2.10) outside the plasma layer ($z = 0$) when the functions $\mathrm{e}^{\mathbf{iks}}\Phi(z)$ and $\mathrm{e}^{\mathbf{iks}}\Psi(z)$ are the eigenfunctions of the operator $(-\Delta)$ with the eigenvalues ω^2/c^2 (Δ is the three-dimensional Laplace operator):

$$-\Delta\, \mathrm{e}^{\mathbf{iks}}\Phi(z) = \frac{\omega^2}{c^2}\,\mathrm{e}^{\mathbf{iks}}\Phi(z), \tag{3.8}$$

$$-\Delta\, \mathrm{e}^{\mathbf{iks}}\Psi(z) = \frac{\omega^2}{c^2}\,\mathrm{e}^{\mathbf{iks}}\Psi(z). \tag{3.9}$$

We deduce from here

$$-\Phi''(z) = \left(\frac{\omega^2}{c^2} - k^2\right)\Phi(z), \tag{3.10}$$

$$-\Psi''(z) = \left(\frac{\omega^2}{c^2} - k^2\right)\Psi(z), \tag{3.11}$$

$$-\infty < z < \infty, \qquad z \neq 0, \qquad 0 \leqslant k^2 < \infty.$$

The substitution of the fields (3.4)–(3.7) into the matching conditions (2.11) and (2.12) gives the analogous conditions at the point $z = 0$ for the functions $\Phi(z)$ and $\Psi(z)$,

$$[\Phi(0)] = -2q\left(\frac{c}{\omega}\right)^2\Phi'(0), \qquad [\Phi'(0)] = 0, \tag{3.12}$$

$$[\Psi(0)] = 0, \qquad [\Psi'(0)] = 2q\Psi(0), \tag{3.13}$$

where the following notation is introduced:

$$[F(z)] = F(z+0) - F(z-0). \tag{3.14}$$

The matching conditions (3.12) are very specific. They involve the eigenvalues of the whole initial spectral problem (3.8). It implies, specifically, that the dynamics of TM-modes along the z-axis 'feels' the parallel dimensions (x, y). Thus for the TM-modes the initial spectral problem does not factorizes completely into two independent boundary value problems along the z-axis and in parallel directions.

In order to complete the formulation of the boundary spectral problems (3.10)–(3.13) we have to specify the behaviour of the functions $\Phi(z)$ and $\Psi(z)$ when $|z| \to \infty$. Proceeding from the physical content of the problem in question, we shall consider such functions $\Phi(z)$ and $\Psi(z)$ which either oscillate ($p^2 \equiv \omega^2/c^2 - k^2 > 0$) or go down ($\kappa^2 \equiv k^2 - \omega^2/c^2 > 0$) when $|z| \to \infty$. In the first case, we deal with the scattering states and in the second one the solutions to the Maxwell equations describe the surface plasmon[3].

The spectral problem for the TE-modes (3.11) and (3.13) with the respective boundary conditions is a self-adjoint spectral problem. Substituting the TE-fields (3.4) and (3.5) into the initial Maxwell equations with δ-sources (2.2) we arrive at the equation for $\Psi(z)$ with the δ-potential (instead of equation (3.11))

$$-\Psi''(z) + 2q\delta(z)\Psi(z) = p^2\Psi(z), \qquad \frac{\omega^2}{c^2} = k^2 + p^2. \tag{3.15}$$

Integration of this equation in vicinity of the point $z = 0$ leads to the matching conditions (3.13) (see, for example, [21]). The potential in equation (3.15) is equal to $2q\delta(z)$. As mentioned

[3] In electrodynamics of continuous media the surface plasmon is a special solution to the Maxwell equations which describes the electric and magnetic fields exponentially decreasing in the direction normal to the interface between two material media [16–18]. The plasmon solution exists only in the TM-sector. It is these configurations of an electromagnetic field that contribute to the Casimir force between two semi-infinite material media [19, 20].

above, the spectral problem for the TM-modes (3.10), (3.12) possesses a peculiarity, namely, the matching conditions (3.12) involve the eigenvalue ω/c. Obviously, the self-adjointness condition is not satisfied here.

The scattering problem for the functions $\Phi(z)$ and $\Psi(z)$ is formulated on a line: $-\infty < z < \infty$. As is known [22, 23], the one-dimensional scattering problem has features in comparison with the description of the scattering processes in three-dimensional space.

The scattering states for the TE-modes are described by the functions

$$\Psi(z) = C_1\,\mathrm{e}^{\mathrm{i}pz} + C_2\,\mathrm{e}^{-\mathrm{i}pz}, \qquad z < 0, \tag{3.16}$$

$$\Psi(z) = C_3\,\mathrm{e}^{\mathrm{i}pz}, \qquad z > 0, \qquad p > 0, \qquad p^2 = \frac{\omega^2}{c^2} - k^2. \tag{3.17}$$

The initial wave is coming from the left. Matching conditions (3.13) give the following relation between the constants $C_i\,(i = 1, 2, 3)$:

$$C_1 + C_2 = C_3, \tag{3.18}$$

$$\mathrm{i}p(C_3 - C_1 + C_2) = 2qC_3. \tag{3.19}$$

From here we obtain for the reflection and transmission coefficients

$$\mathcal{R}^{\mathrm{TE}} \equiv \frac{C_2}{C_1} = \frac{-\mathrm{i}q}{p + \mathrm{i}q} = \mathrm{i}\sin\eta\,\mathrm{e}^{\mathrm{i}\eta}, \qquad \mathcal{T}^{\mathrm{TE}} = \frac{p}{p + \mathrm{i}q} = \cos\eta\,\mathrm{e}^{\mathrm{i}\eta}, \tag{3.20}$$

where $\eta(p)$ is the phase shift for the scattering of the TE-modes. When initial waves are coming from the right we obtain the same reflection and transition coefficients. There are no other conditions on p besides the positivity $p > 0$ (continuous spectrum). The scattering matrix, $S(p)$, is determined by the phase shift $\eta(p)$,

$$\tan\eta(p) = -\frac{p}{q}, \qquad p > 0, \tag{3.21}$$

in a standard way

$$S(p) = \mathrm{e}^{2\mathrm{i}\eta(p)}. \tag{3.22}$$

Between TE-modes there are no solutions which go down when $|z| \to \infty$. Indeed, such solutions, in accord with equation (3.11) would have the form

$$\begin{aligned} &\Psi(z) = C_1\,\mathrm{e}^{\kappa z}, \qquad z < 0, \\ &\Psi(z) = C_2\,\mathrm{e}^{-\kappa z}, \qquad z > 0, \qquad \kappa = +\sqrt{k^2 - \frac{\omega^2}{c^2}} > 0. \end{aligned} \tag{3.23}$$

Substitution of equation (3.23) into the matching conditions (3.13) gives

$$C_1 = C_2 \qquad \text{and} \qquad \kappa = -q < 0. \tag{3.24}$$

Thus, there are no solutions of the form (3.23) with positive κ. This is in accord with the known fact [21] that the potential defined by the Dirac δ-function $\lambda\delta(z)$ leads to the bound state only for $\lambda < 0$. In our case $\lambda = 2q > 0$ (see equation (3.15)).

Finally, the TE-modes have the spectrum

$$\frac{\omega^2(\mathbf{k}, p)}{c^2} = k^2 + p^2, \qquad \mathbf{k} \in \mathbb{R}^2, \qquad 0 \leqslant p < \infty. \tag{3.25}$$

Here the contribution k^2 is given by the free waves propagating in directions parallel to the plasma layer, and the contribution p^2 corresponds to the one-dimensional scattering in normal direction with the phase shift (3.21).

Proceeding in the same way one obtains for the scattering of the TM-modes

$$\mathcal{R}^{\mathrm{TM}} = \frac{\mathrm{i}pq}{p^2+k^2+\mathrm{i}pq} = -\mathrm{i}\sin\mu\,\mathrm{e}^{\mathrm{i}\mu}, \qquad \mathcal{T}^{\mathrm{TM}} = \frac{p^2+k^2}{p^2+k^2+\mathrm{i}pq} = \cos\mu\,\mathrm{e}^{\mathrm{i}\mu}, \tag{3.26}$$

where the phase shift $\mu(p,k)$ is defined by

$$\tan\mu(p,k) = -\frac{pq}{p^2+k^2}, \qquad p \geqslant 0. \tag{3.27}$$

The scattering matrix is defined here by the formula analogous to equation (3.22): $S(p,k) = \mathrm{e}^{2\mathrm{i}\mu(p,k)}$.

It turns out that between TM-modes there are solutions which not only oscillate when $|z| \to \infty$, but also go down in this limit. In this case, the function $\Phi(z)$ is defined by the equation

$$\Phi''(z) - \left(k^2 - \frac{\omega^2}{c^2}\right)\Phi(z) = 0, \tag{3.28}$$

where

$$k^2 - \frac{\omega^2}{c^2} \equiv \kappa^2 > 0. \tag{3.29}$$

The solution we are interested in should have the form

$$\begin{aligned} \Phi(z) &= C_1\,\mathrm{e}^{\kappa z}, \qquad z < 0,\\ \Phi(z) &= C_2\,\mathrm{e}^{-\kappa z}, \qquad z > 0, \end{aligned} \tag{3.30}$$

where $\kappa = +\sqrt{k^2-\omega^2/c^2} > 0$. The matching conditions (3.12) give rise to the equation for κ,

$$\kappa^2 + \kappa q - k^2 = 0, \tag{3.31}$$

the positive root of which is

$$\kappa = \sqrt{\frac{q^2}{4} + k^2} - \frac{q}{2}. \tag{3.32}$$

For the respective frequency squared we derive by making use of equation (3.29)

$$\frac{\omega^2_{\mathrm{sp}}}{c^2} = \frac{q}{2}\left(\sqrt{q^2+4k^2} - q\right) \geqslant 0. \tag{3.33}$$

Thus, the frequency of the surface plasmon is real, and the solution obtained oscillates in time instead of damping.

Finally, the spectrum of the TM-modes in the problem under study have two branches: (i) the photon branch and (ii) the surface plasmon branch. The photon spectrum of TM-modes is defined by

$$\frac{\omega^2_{\mathrm{ph}}(\mathbf{k},p)}{c^2} = \mathbf{k}^2 + p^2, \qquad \mathbf{k} \in \mathbb{R}^2, \quad 0 \leqslant p < \infty. \tag{3.34}$$

Here the contribution $\mathbf{k}^2$, as in equation (3.25) is given by the free waves propagating in directions parallel to the plasma layer and p^2 is the contribution of the one-dimensional scattering with the phase shift (3.27) in normal (to the plasma sheet) direction. The surface plasmon branch of the spectrum is described by

$$\frac{\omega^2_{\mathrm{sp}}(\mathbf{k})}{c^2} = \frac{q}{2}\left(\sqrt{q^2+4k^2} - q\right), \qquad \mathbf{k} \in \mathbb{R}^2. \tag{3.35}$$

In the spectral problem (3.10), (3.12) with negative q the parameter κ defined by equation (3.32) is positive, however the respective frequency squared is negative

$$\frac{\omega^2_{q<0}}{c^2} = -\frac{1}{2}\left(q^2 + |q|\sqrt{q^2+4k^2}\right). \tag{3.36}$$

Hence, in this case we have a resonance state instead of the bound state (surface plasmon).

4. Spectral functions in a flat plasma sheet model

In the previous section, we have determined the spectrum in the model under consideration (see equations (3.25), (3.34) and (3.35)) therefore we can proceed with the construction of the spectral functions in this model; namely, the spectral zeta-function

$$\zeta(s) = \mathrm{Tr}\, L^{-s} = \sum_n \lambda_n^{-s} \tag{4.1}$$

and (integrated) heat kernel

$$K(t) = \mathrm{Tr}(\mathrm{e}^{-tL}) = \sum_n \mathrm{e}^{-\lambda_n t}, \tag{4.2}$$

where L is the differential operator in equations (3.10) and (3.11) with matching conditions at $z = 0$ (3.12), (3.13) and conditions specified above for $|z| \to \infty$; λ_n are the eigenvalues given by equations (3.25), (3.34) and (3.35). In equation (4.2) t is an auxiliary variable, $0 \leqslant t < \infty$. In our case $L = -\Delta$ and t has the dimension [length]^2.

As a preliminary, it is worth stipulating the procedure of spectral summation entered equations (4.1) and (4.2). Obviously, the contribution to the spectral functions generated by free waves is given by a simple integration over the respective wave vector $\mathbf{k}$:

$$\int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \cdots = \int_0^\infty \frac{k\,\mathrm{d}k}{2\pi} \cdots. \tag{4.3}$$

We define the contribution of the scattering states by making use of the phase shift method; namely, the integration over dp will be carried out with the spectral density $\rho(p)$ (the function of spectral shift),

$$\rho(p) = \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}p} \ln S(p) = \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}p} \delta(p), \tag{4.4}$$

where $\delta(p)$ is the phase shifts for TE- and TM-modes given by equations (3.21) and (3.27), respectively.

4.1. TE-modes

Taking into account all this we get, for the spectral zeta-function in the TE-sector of the model,

$$\zeta^{\mathrm{TE}}(s) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \int_0^\infty \mathrm{d}p (k^2 + p^2)^{-s} \frac{1}{\pi} \frac{\mathrm{d}}{\mathrm{d}p} \eta(p) = \frac{q}{2\pi^2} \int_0^\infty \frac{\mathrm{d}p}{p^2 + q^2} \int_0^\infty \frac{k\,\mathrm{d}k}{(k^2 + p^2)^s}. \tag{4.5}$$

The left-hand side of equation (4.5) is defined in the semi-plane Re $s > 1$ and this representation can be analytically extended all over the complex plane s saved for separate points. In order to do this, it is sufficient to express the integrals in equation (4.5) in terms of the gamma-function

$$\zeta^{\mathrm{TE}}(s) = \frac{q^{2-2s}}{8\pi^2} \frac{\Gamma(3/2 - s)\Gamma(s - 1/2)\Gamma(s - 1)}{\Gamma(s)} = \frac{q^{2-2s}}{8\pi(1 - s)\cos(\pi s)}. \tag{4.6}$$

The integrated heat kernel for TE-modes is given by

$$K^{\mathrm{TE}}(t) = K_0^{(d=2)}(t) \cdot K^{(d=1)}(t), \tag{4.7}$$

where $K_0^{(d=2)}(t)$ is the free two-dimensional heat kernel responsible for free waves propagating in directions parallel to the plasma layer

$$K_0^{(d=2)}(t) = \int_0^\infty \frac{k\,\mathrm{d}k}{2\pi} \mathrm{e}^{-k^2 t} = \frac{1}{4\pi t}, \tag{4.8}$$

and

$$K^{(d=1)}(t) = \frac{1}{\pi}\int_0^\infty \mathrm{d}p\, \mathrm{e}^{-p^2 t}\frac{\mathrm{d}}{\mathrm{d}p}\arctan\left(-\frac{q}{p}\right) = \frac{q}{\pi}\int_0^\infty \frac{\mathrm{e}^{-p^2 t}}{p^2+q^2}\,\mathrm{d}p$$
$$= \frac{1}{2}\,\mathrm{e}^{tq^2}[1-\mathrm{erf}(q\sqrt{t})] = \frac{1}{2}\,\mathrm{e}^{tq^2}\mathrm{erfc}(q\sqrt{t}), \tag{4.9}$$

where $\mathrm{erf}(q\sqrt{t})$ is the probability integral [24, 25].

In the plasma sheet model under consideration the parameter q is strictly positive. However, the spectral problem (3.15) can be considered for the negative q also when it possesses a bound state [21]. The contribution of this state should be taken into account when calculating the respective heat kernel $K_{q<0}(t)$. But this heat kernel can be obtained by analytical continuation of equation (4.9)

$$K_{q<0}(t) = \tfrac{1}{2}\,\mathrm{e}^{tq^2}[1-\mathrm{erf}(-|q|\sqrt{t})] = \tfrac{1}{2}\,\mathrm{e}^{tq^2}[1+\mathrm{erf}(|q|\sqrt{t})]. \tag{4.10}$$

Comparison equations (4.9) and (4.10) give the following simple relation between the integrated heat kernel for δ-potentials with positive and negative coupling constants

$$K_{q>0}(t) + K_{q<0}(t) = \mathrm{e}^{tq^2}, \tag{4.11}$$

where $K_{q>0}(t)$ is given by equation (4.9).

The structure of divergences in quantum field theory is determined by the coefficients of the asymptotic expansion of the heat kernel when $t \to +0$,

$$K(t) = (4\pi t)^{-d/2}\sum_{n=0,1,2,\ldots} t^{n/2}B_{n/2} + \mathrm{ES}, \tag{4.12}$$

where d is the dimension of the configuration space in the problem at hand, ES stands for the exponentially small corrections. The coefficients $B_{n/2}$ for TE-modes can be derived in two different ways: (i) by making use of equations (4.7)–(4.9) which determine the heat kernel explicitly and (ii) by applying the known relation between the residua of the product $\zeta(s)\Gamma(s)$ and $B_{n/2}$:

$$\frac{B_{n/2}}{(4\pi)^{d/2}} = \lim_{s\to\frac{d-n}{2}}\left(s+\frac{n-d}{2}\right)\Gamma(s)\zeta(s), \qquad n = 0, 1, 2, \ldots. \tag{4.13}$$

Both calculation schemes give the same values for the coefficients $B_{n/2}$ with $n \neq 0$:

$$B_{1/2} = \sqrt{\pi}, \qquad B_1 = -2q, \qquad B_{3/2} = \sqrt{\pi}q^2, \qquad B_2 = -\frac{4}{3}q^3,$$
$$B_{5/2} = \frac{\sqrt{\pi}}{2}q^4, \qquad B_3 = -\frac{8}{15}q^5, \qquad B_{7/2} = \frac{\sqrt{\pi}}{6}q^6, \qquad \ldots. \tag{4.14}$$

Equations (4.7)–(4.9) give for the coefficient B_0

$$B_0 = 0, \tag{4.15}$$

while employment of the zeta-function (4.5) and relation (4.13) leads to the nonzero value

$$B_0 = -\frac{1}{q}. \tag{4.16}$$

The different values of B_0 in equations (4.15) and (4.16) are the consequence of the noncompact configuration space, $\mathbb{R}^3$, in the problem at hand. In the case of compact manifolds the coefficient B_0 is equal to the volume V of the manifold: $B_0 = V$. Definition of B_0 for a noncompact configuration space requires a special consideration [27, 26]. The correct value of B_0 in our case is formula (4.15) derived from the explicit heat kernel (4.7)–(4.9). The point is, the contribution of unbounded space $\mathbb{R}^3$ is subtracted from the spectral density (4.4). Hence this contribution is subtracted also from the multiplier $K^{(d-1)}(t)$ in equation (4.7) and, due to the factorization of this formula, the contribution of $\mathbb{R}^3$ is automatically subtracted from the total kernel $K(t)$.

When calculating the zeta-function $\zeta^{\mathrm{TE}}(s)$ the spectral density (4.4) with subtracted $\mathbb{R}^3$ contribution has also been used. However in equation (4.5), unlike equation (4.7), there is no factorization of the contributions of waves propagating in directions parallel to the plasma plane and in the direction normal to this plane. It is this point that leads to a nonzero value of B_0 in equation (4.16).

4.2. TM-modes

In the case of TM-modes there are two branches of the spectrum: (i) the photon branch (3.34) and (ii) the surface plasmon branch (3.35).

By making use of equations (3.27) and (4.4) we obtain for the photon zeta-function in the TM-sector of the model

$$\begin{aligned}\zeta_{\mathrm{ph}}^{\mathrm{TM}}(s) &= \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\int_0^\infty \mathrm{d}p\,(k^2+p^2)^{-s}\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}p}\mu(p,k)\\ &= \frac{q}{2\pi^2}\int_0^\infty k\,\mathrm{d}k\int_0^\infty \frac{\mathrm{d}p}{(k^2+p^2)^s}\,\frac{p^2-k^2}{(k^2+p^2)^2+q^2p^2}.\end{aligned} \tag{4.17}$$

The integration in (4.17) can be easily done by introducing first the polar coordinates

$$\begin{aligned} &k = r\sin\varphi, \qquad p = r\cos\varphi, \qquad \mathrm{d}k\,\mathrm{d}p = r\,\mathrm{d}r\,\mathrm{d}\varphi,\\ &0 \leqslant \varphi \leqslant \pi/2, \qquad 0 \leqslant r \leqslant \infty.\end{aligned} \tag{4.18}$$

In terms of new variables, formula (4.17) acquires the form

$$\zeta_{\mathrm{ph}}^{\mathrm{TM}}(s) = \frac{q}{2\pi^2}\int_0^{\pi/2}\mathrm{d}\varphi\int_0^\infty \mathrm{d}r\,\frac{r^{2-2s}\sin\varphi(2\cos^2\varphi-1)}{r^2+q^2\cos^2\varphi}. \tag{4.19}$$

Now it is natural to make the following substitution of the variables:

$$\cos\varphi = x, \qquad -\sin\varphi\,\mathrm{d}\varphi = \mathrm{d}x. \tag{4.20}$$

As a result one gets

$$\zeta_{\mathrm{ph}}^{\mathrm{TM}}(s) = \frac{q}{2\pi^2}\int_0^1 \mathrm{d}x\,(2x^2-1)\int_0^\infty \frac{r^{2-2s}\,\mathrm{d}r}{r^2+q^2x^2}. \tag{4.21}$$

The integral over r in (4.21) exists in the region

$$1/2 < \mathrm{Re}\,s < 3/2, \tag{4.22}$$

and the subsequent integration over x can be done for

$$1/2 < \mathrm{Re}\,s < 1. \tag{4.23}$$

Thus the integral representation (4.21) can be analytically extended all over the complex s plane. This yields

$$\zeta_{\mathrm{ph}}^{\mathrm{TM}}(s) = \frac{q^{2-2s}}{8\pi^2}\,\frac{s}{(s-1)(2-s)}\Gamma(3/2-s)\Gamma(s-1/2) = \frac{q^{2-2s}}{8\pi}\,\frac{s}{(1-s)(2-s)\cos\pi s}. \tag{4.24}$$

Let us proceed with the construction of the heat kernel for the photon branch of the spectrum in the TM-sector of the model at hand. By making use of equations (3.34), (4.2), (4.3) and (4.4), we obtain

$$K_{\mathrm{ph}}^{\mathrm{TM}}(t) = \frac{q}{2\pi^2}\int_0^\infty k\,\mathrm{d}k\,\mathrm{e}^{-k^2t}\int_0^\infty \mathrm{d}p\,\mathrm{e}^{-p^2t}\frac{p^2-k^2}{(k^2+p^2)^2+q^2p^2}. \tag{4.25}$$

Further, we again introduce the polar coordinates (4.18) and then make the change of variables (4.20). Finally, calculations are reduced to the substitution of the term r^{-2s} in equation (4.21) by $\mathrm{e}^{-r^2 t}$,

$$K_{\mathrm{ph}}^{\mathrm{TM}}(t) = \frac{q}{2\pi^2}\int_0^1 \mathrm{d}x\,(2x^2-1)I(x), \tag{4.26}$$

where

$$\begin{aligned} I(x) &= \int_0^\infty \frac{r^2\,\mathrm{e}^{-r^2 t}}{r^2+q^2x^2}\,\mathrm{d}r = \frac{1}{2}\sqrt{\frac{\pi}{t}} - \frac{\pi q x}{2}\,\mathrm{e}^{q^2x^2t}[1-\mathrm{erf}(qx\sqrt{t})] \\ &= \frac{1}{2}\sqrt{\frac{\pi}{t}} - \frac{\pi q x}{2}\left[\mathrm{e}^{q^2x^2t} - \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{2^k(qx\sqrt{t})^{2k+1}}{(2k+1)!!}\right]. \end{aligned} \tag{4.27}$$

We have used here the series representation for the function $\mathrm{erf}(qx\sqrt{t})$ [24, 25].

Equations (4.13) and (4.24), on the one hand, and the explicit form of the heat kernel, equations (4.26) and (4.27), on the other hand, give the same value for the coefficients $B_{n/2}$ in expansion (4.12) for all n saved for $n = 0, 1$:

$$\begin{aligned} &B_1 = -\frac{2}{3}q, && B_{3/2} = 0, && B_2 = \frac{4}{15}q^3, \\ &B_{5/2} = -\frac{\sqrt{\pi}}{6}q^4, && B_3 = \frac{8}{35}q^5, && B_{7/2} = -\frac{\sqrt{\pi}}{12}q^6. \end{aligned} \tag{4.28}$$

The explicit form of the integrated heat kernel in question, (4.26) and (4.27), shows that first two coefficients in expansion (4.12) vanish while formulae (4.26) and (4.27) give

$$B_0 = -\frac{3}{q}, \qquad B_{1/2} = \sqrt{\pi}. \tag{4.29}$$

This discrepancy implies that calculation of the first heat kernel coefficients through the respective spectral zeta-function requires a special consideration in the case of unbounded configuration space.

Let us address now the spectral functions for the spectrum branch (3.35) generated by the surface plasmon. Equations (3.35), (4.1) and (4.3) give

$$\zeta_{\mathrm{sp}}^{\mathrm{TM}}(s) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\frac{\omega_{\mathrm{sp}}^{-2s}(\mathbf{k})}{c^{-2s}} = \left(\frac{2}{q}\right)^s \int_0^\infty \frac{k\,\mathrm{d}k}{2\pi}\left(\sqrt{q^2+4k^2}-q\right)^{-s}. \tag{4.30}$$

The convergence of this integral in the region $k \to 0$ requires $\mathrm{Re}\,s < 1$, but when $k \to \infty$ it exists only if $\mathrm{Re}\,s > 2$. Thus for the spectrum (3.35) it is impossible to construct the zeta-function by making use of the analytical continuation method, since one cannot define this function in any finite domain of the complex plane s.

However, it turns out that for this branch of the spectrum the heat kernel can be constructed explicitly. Indeed, by making use of equations (4.2), (4.3) and (3.35) we obtain

$$K_{\mathrm{sp}}^{\mathrm{TM}}(t) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2}\exp\left[-\frac{\omega_{\mathrm{sp}}^2(\mathbf{k})}{c^2}t\right] = \int_0^\infty \frac{k\,\mathrm{d}k}{2\pi}\exp\left[-\frac{q}{2}\left(\sqrt{q^2+4k^2}-q\right)t\right]. \tag{4.31}$$

The change of variables

$$\frac{q}{2}\left(\sqrt{q^2+4k^2}-q\right) = x, \qquad k\,\mathrm{d}k = \left(\frac{x}{q^2}+\frac{1}{2}\right)\mathrm{d}x$$

reduces the integral (4.31) to the form

$$K_{\mathrm{sp}}^{\mathrm{TM}}(t) = \int_0^\infty \frac{\mathrm{d}x}{2\pi}\left(\frac{x}{q^2}+\frac{1}{2}\right)\mathrm{e}^{-tx} = \frac{1}{2\pi q^2 t^2} + \frac{1}{4\pi t}. \tag{4.32}$$

The first term in equation (4.32) is absent in the standard expansion (4.12), and the second term in (4.32) yields $B_{1/2} = 2\sqrt{\pi}$. The rest of coefficients $B_{n/2}$ with $n \neq 1$ equal zero. Thus the surface plasmon with the spectrum (3.35) is a simple (and at the same time, of a physical meaning) model that has no spectral zeta-function, but the respective (integrated) heat kernel exists though with a nonstandard asymptotic expansion. It is not surprising because we are dealing here with a singular point interaction described by the matching conditions (3.12).

5. Local heat kernel

In the TE-sector of the model under consideration one can construct, in an explicit form, the local heat kernel or Green function

$$K(\mathbf{r}, \mathbf{r}'; t) = \langle \mathbf{r}| \, \mathrm{e}^{-tL} |\mathbf{r}'\rangle = \sum_n \varphi_n^*(\mathbf{r})\varphi_n(\mathbf{r}') \, \mathrm{e}^{-\lambda_n t}. \tag{5.1}$$

We are using here the same notation as in equation (4.2) and $\varphi_n(\mathbf{r})$ are normalized eigenfunctions in the spectral problems at hand

$$L\varphi_n(\mathbf{r}') = \lambda_n \varphi_n(\mathbf{r}') \qquad L = -\Delta. \tag{5.2}$$

From the definition (5.1) it follows, in particular, that the local heat kernel obeys the heat conduction equations with respect to the variables $(\mathbf{r}, t)$ and $(\mathbf{r}', t)$

$$\left(\Delta_{\mathbf{r}} - \frac{\partial}{\partial t}\right) K(\mathbf{r}, \mathbf{r}'; t) = 0, \tag{5.3}$$

$$\left(\Delta_{\mathbf{r}'} - \frac{\partial}{\partial t}\right) K(\mathbf{r}, \mathbf{r}'; t) = 0 \tag{5.4}$$

and initial condition

$$K(\mathbf{r}, \mathbf{r}'; t) \to \delta(\mathbf{r} - \mathbf{r}'), \qquad \text{when} \quad t \to 0^+. \tag{5.5}$$

Obviously, in the case of TE-modes $K(\mathbf{r}, \mathbf{r}'; t)$ can be represented in a factorized form

$$K(\mathbf{r}, \mathbf{r}'; t) = K_0^{(d=2)}(\mathbf{s}, \mathbf{s}'; t) \cdot K(z, z'; t), \qquad \mathbf{r} = (\mathbf{s}, z), \qquad \mathbf{s} = (x, y), \tag{5.6}$$

where $K_0^{(d=2)}(\mathbf{s}, \mathbf{s}'; t)$ is the free heat kernel in directions parallel to the plane $z = 0$:

$$K_0^{(d=2)}(\mathbf{s}, \mathbf{s}'; t) = \frac{1}{4\pi t} \exp\left[-\frac{(\mathbf{s} - \mathbf{s}')^2}{4t}\right]. \tag{5.7}$$

For constructing the heat kernel $K(z, z'; t)$ along the infinite z-axes we shall use the technique of integral equations developed in our previous paper [28]. These equations naturally arise when representing the Green function to be found in terms of heat potentials of simple and double layers. With respect of its first argument z the heat kernel $K(z, z'; t)$ obeys the matching conditions (3.13) and when $t \to 0^+$ it satisfies the initial condition (5.5).

In what follows, it is convenient to represent the heat kernel we are looking for in terms of four components depending on the range of its arguments,

$$K(z, z'; t) = \begin{cases} K_{-+}(z, z'; t), & z < 0, \quad z' > 0, \\ K_{++}(z, z'; t), & z, z' > 0, \\ K_{+-}(z, z'; t), & z > 0, \quad z' < 0, \\ K_{--}(z, z'; t), & z, z' < 0. \end{cases} \tag{5.8}$$

We represent the heat kernel $K(z, z'; t)$ in terms of heat potentials of simple layers [28]

$$K_{-+}(z, z'; t) = \int_0^t \mathrm{d}\tau\, K_0(z, 0; t-\tau)\alpha_1(\tau; z'), \qquad z < 0, \quad z' > 0, \tag{5.9}$$

$$K_{++}(z, z'; t) = K_0(z, z'; t) + \int_0^t \mathrm{d}\tau\, K_0(z, 0; t-\tau)\alpha_2(\tau; z'), \qquad z, z' > 0, \tag{5.10}$$

where $\alpha_1(\tau; z')$ and $\alpha_2(\tau; z')$ are the densities of the heat potentials to be found and $K_0(z, z'; t)$ is the free heat potential (Green's function) on an infinite line

$$K_0(z, z'; t) = \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{(z-z')^2}{4t}\right]. \tag{5.11}$$

Substituting equations (5.9) and (5.10) in matching conditions (3.13) we obtain

$$\int_0^t \mathrm{d}\tau\, K_0(0, 0; t-\tau)[\alpha_1(\tau; z') - \alpha_2(\tau; z')] = K_0(0, z'; t), \qquad z' > 0, \tag{5.12}$$

$$2\frac{\partial K_0}{\partial z}(z = 0, z'; t) - 4q\int_0^t \mathrm{d}\tau\, K_0(0, 0; t-\tau)\alpha_1(\tau; z') = \alpha_1(\tau; z') + \alpha_2(\tau; z'), \qquad z' > 0. \tag{5.13}$$

When deriving equation (5.13) we have taken into account that the derivative of the single layer potential has a jump at the interface; namely, let $V(z; t)$ be the heat potential of a single layer with the density $\nu(\tau)$,

$$V(z; t) = \int_0^t \mathrm{d}\tau\, K_0(z, 0; t-\tau)\nu(\tau), \tag{5.14}$$

then one can easily show that [28, 29]

$$\frac{\partial V}{\partial z}(z = 0^+; t) = -\frac{1}{2}\nu(t), \qquad \frac{\partial V}{\partial z}(z = 0^-; t) = \frac{1}{2}\nu(t). \tag{5.15}$$

On substituting in equation (5.12) the explicit form of the free heat kernel (5.11) we arrive at the Abel integral equation, the exact solution of which is known [30, 31], or can be derived by the Laplace transform

$$\alpha_1(t; z) - \alpha_2(t; z) = \frac{z}{t\sqrt{4\pi t}} \exp\left(-\frac{z^2}{4t}\right), \qquad z > 0. \tag{5.16}$$

In what follows, it is convenient to apply to equations (5.13) and (5.16) the Laplace transform

$$\bar{f}(p) = \int_0^\infty \mathrm{e}^{-pt} f(t)\, \mathrm{d}t. \tag{5.17}$$

As a result, we obtain the set of two linear algebraic equations for the functions $\bar{\alpha}_1(p; z')$ and $\bar{\alpha}_2(p; z')$:

$$\bar{\alpha}_1(p; z') - \bar{\alpha}_2(p; z') = \mathrm{e}^{-z'\sqrt{p}}, \qquad z' > 0, \tag{5.18}$$

$$\bar{\alpha}_1(p; z') + \bar{\alpha}_2(p; z') = \mathrm{e}^{-z'\sqrt{p}} - \frac{2q}{\sqrt{p}}\bar{\alpha}_1(p; z'), \qquad z' > 0. \tag{5.19}$$

From here we deduce

$$\bar{\alpha}_1(p; z') = \frac{\sqrt{p}}{q + \sqrt{p}}\, \mathrm{e}^{-z'\sqrt{p}}, \qquad \bar{\alpha}_2(p; z') = -\frac{q}{q + \sqrt{p}}\, \mathrm{e}^{-z'\sqrt{p}}, \qquad z' > 0. \tag{5.20}$$

By making use of the Laplace transform (5.17) the initial formulae (5.9) and (5.10) can be rewritten in the form

$$\bar{K}_{-+}(z,z';p)=\frac{1}{2\sqrt{p}}\,\mathrm{e}^{-|z|\sqrt{p}}\bar{\alpha}_1(p;z'),\qquad z<0,\quad z'>0,\tag{5.21}$$

$$\bar{K}_{++}(z,z';p)=\frac{1}{2\sqrt{p}}\,\mathrm{e}^{-|z-z'|\sqrt{p}}+\frac{1}{2\sqrt{p}}\,\mathrm{e}^{-z\sqrt{p}}\bar{\alpha}_2(p;z'),\qquad z,z'>0.\tag{5.22}$$

Substitution of equation (5.20) into equations (5.21) and (5.22) yields

$$\bar{K}_{-+}(z,z';p)=\frac{1}{2(q+\sqrt{p})}\,\mathrm{e}^{-(|z|+z')\sqrt{p}},\qquad z<0,\quad z'>0,\tag{5.23}$$

$$\bar{K}_{++}(z,z';p)=\frac{1}{2\sqrt{p}}\,\mathrm{e}^{-|z-z'|\sqrt{p}}-\frac{q}{2\sqrt{p}(q+\sqrt{p})}\,\mathrm{e}^{-(z+z')\sqrt{p}},\qquad z,z'>0.\tag{5.24}$$

The inverse Laplace transform [25, 32] applied to equations (5.23) and (5.24) gives

$$K_{-+}(z,z';t)=K_0(|z|+z',0;t)-\frac{q}{2}\,\mathrm{e}^{q(|z|+z')+q^2t}\mathrm{erfc}\left(q\sqrt{t}+\frac{|z|+z'}{2\sqrt{t}}\right),\qquad z<0,\quad z'>0,\tag{5.25}$$

$$K_{++}(z,z';t)=K_0(z,z';t)-\frac{q}{2}\,\mathrm{e}^{q(z+z')+q^2t}\mathrm{erfc}\left(q\sqrt{t}+\frac{z+z'}{2\sqrt{t}}\right),\qquad z,z'>0.\tag{5.26}$$

By making use of the same technique one can derive for the components $K_{+-}(z,z';t)$ and $K_{--}(z,z';t)$ the formulae which are analogous to equations (5.25) and (5.26). It turns out that all four components of the heat kernel $K(z,z';t)$ (see equation (5.8)) can be represented in a unique way, namely,

$$K(z,z';t)=K_0(z,z';t)-\frac{q}{2}\,\mathrm{e}^{q(|z|+|z'|)+q^2t}\mathrm{erfc}\left(q\sqrt{t}+\frac{|z|+|z'|}{2\sqrt{t}}\right),$$
$$-\infty<z,\quad z'<\infty,\quad z\neq 0,\quad z'\neq 0.\tag{5.27}$$

In the literature [33, 34] the fundamental solutions to the heat conduction equation or to the Schrödinger equation with the Dirac δ-potential are represented, as a rule, in an integral form

$$K(z,z';t)=K_0(z,z';t)-q\int_0^\infty \mathrm{e}^{-qu}K_0(|z|+|z'|+u,0;t)\,\mathrm{d}u.\tag{5.28}$$

It is worth noting that the formulae (5.27) and (5.28) have the sense also at the points $z=0$ and $z'=0$.

Thus the technique of integral equations developed on the basis of heat potentials in [28] affords a regular and simple method for deriving, in an explicit form, the heat kernel in the problem under consideration. In [35] the representation (5.27) was obtained in the framework of another approach to this problem.

Proceeding from equation (5.28) one can easily obtain the traced heat kernel (4.9) by evaluating the integral

$$K(t)=\int_{-\infty}^{\infty}\mathrm{d}z\,[K(z,z;t)-K_0(z,z;t)]$$
$$=-q\int_{-\infty}^{\infty}\mathrm{d}z\int_0^\infty \mathrm{d}u\,\mathrm{e}^{-qu}K_0(2|z|+u,0;t).\tag{5.29}$$

Substitution to this equation of the exact form of the free heat kernel (5.11) gives

$$K(t) = -\frac{q}{\sqrt{\pi t}}\int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}u\, \mathrm{e}^{-qu} \exp\left[-\frac{(2z+u)^2}{4t}\right]$$
$$= -\frac{q}{2}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-qu}\,\mathrm{erfc}\left(\frac{u}{2\sqrt{t}}\right). \tag{5.30}$$

Thus, in order to find the integrated heat kernel $K(t)$ one has to calculate the Laplace transform of the function $\mathrm{erfc}(u/(2\sqrt{t}))$. By making use of the table of the Laplace transform [32] one finds

$$K(t) = \tfrac{1}{2}\,\mathrm{e}^{tq^2}\mathrm{erfc}(q\sqrt{t}) - \tfrac{1}{2}. \tag{5.31}$$

The first term on the right-hand side of equation (5.31) exactly reproduces equation (4.9) while the constant term $(-1/2)$ should be removed by introducing appropriate renormalization when calculating the trace over the continuous variables.

In the TM-sector of the model under study the construction of the local heat kernel proves to be much more complicated. First of all, the factorization equation (5.6) does not hold anymore. However, the technique of integral equations [28] can be applied here again, but the pertinent integral equations will be defined on the whole plane $z = 0$; i.e., they are two-dimensional integral equations. In order to go along this line, preliminarily one has to remove the dependence on the spectral parameter $(\omega(p,\mathbf{k})/c)^2$ from the first matching condition (3.12). For the heat kernel $K(\mathbf{r},\mathbf{r};t)$ this condition can be transformed as follows,

$$\left[\frac{\partial K}{\partial t}(\mathbf{s}, z=0, \mathbf{r}'; t)\right] = 2q\frac{\partial K}{\partial z}(\mathbf{s}, z=0, \mathbf{r}'; t), \tag{5.32}$$

where the notations $\mathbf{r} = (\mathbf{s}, z)$, $\mathbf{s} = (x, y)$, and (3.14) are used as before.

The validity of this relation can be checked easily if one takes into account that the eigenfunctions $\varphi(\mathbf{r})\,\mathrm{e}^{-\lambda_n t}$ entering the definition of the heat kernel (5.1) obey equations (5.3) and (5.4)

$$-\Delta\varphi_n(\mathbf{r})\,\mathrm{e}^{-\lambda_n t} = \lambda_n\varphi_n(\mathbf{r})\,\mathrm{e}^{-\lambda_n t} = -\frac{\partial}{\partial t}\varphi_n(\mathbf{r})\,\mathrm{e}^{-\lambda_n t}. \tag{5.33}$$

Here the eigenvalues λ_n are equal to $(\omega(p,\mathbf{k})/c)^2$. The matching condition (5.32) involves the space and time derivatives, i.e., the so-called skew derivative.

The Green function $K(\mathbf{r},\mathbf{r}';t)$ obeys the heat conduction equations (5.3) and (5.4), hence the matching condition (5.32) can be obviously rewritten in the form

$$[\Delta_{\mathbf{r}}K(\mathbf{s}, z=0, \mathbf{r}'; t)] = 2q\frac{\partial K}{\partial z}(\mathbf{s}, z=0, \mathbf{r}'; t). \tag{5.34}$$

An interesting spectral problem arises here when we confine ourselves to the one-dimensional problem with $(\omega(p)/c)^2 = p^2$:

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\varphi(z) = p^2\varphi(z), \qquad -\infty < z < \infty, \quad z \neq 0, \tag{5.35}$$

$$[\varphi'(z=0)] = 0, \tag{5.36}$$

$$[\varphi''(z=0)] = 2q\varphi'(z=0). \tag{5.37}$$

At first sight, this spectral problem is completely different from the analogous one for the δ-potential (see equations (3.11), (3.15) and (3.13)). However, the respective eigenfunctions

$\varphi_p(z)$ and $\Phi_p(z)$ are connected by the relation

$$\varphi_p'(z) = \Phi_p(z). \tag{5.38}$$

Both problems have the same positive continuous spectrum

$$0 < p^2 < \infty. \tag{5.39}$$

Furthermore, the respective phase shifts, and consequently the scattering matrices, coincide (see equations (3.21) and (3.23) with $k = 0$). From here we infer immediately that these spectral problems have the same integrated heat kernels defined by equation (4.9), while the local heat kernels are obviously different.

The local heat kernel for the spectral problem (5.35)–(5.37) can be derived in the same way as has been done above in the spectral problem with δ-potential by making use of the integral equations.

As far as we know, a couple of spectral problems possessing such interesting features are found for the first time.

6. Conclusion

The plasma sheet model investigated here proves to be very interesting and instructive, first of all from the standpoint of spectral analysis. The spectrum of the model contains both continuous branches and bound states (surface plasmon). It is remarkable that for the latter the spectral zeta-function cannot be constructed at least by the standard analytic continuation method. At the same time, the integrated heat kernel is found in an explicit form for all branches of the spectrum. On the whole, this heat kernel has the asymptotic expansion of a noncanonical form due to the singular point interaction.

By making use of the technique of integral equations developed by us earlier [28] the local heat kernel in the TE-sector of the model is found in an explicit form. By the way, here a new method is demonstrated for deriving the fundamental solution to the heat conduction equation (or to the Schrödinger equation) on an infinite line with the δ-like source. In principle, the integral equations technique is also applicable to the construction of the local heat kernel in the TM-sector of the model.

For the heat equation on an infinite line with the δ-source a nontrivial counterpart is found; namely, a spectral problem with point interaction, which possesses the same integrated heat kernel. However, the local heat kernels in these spectral problems are different. The new boundary problem is not self-adjoint, the matching conditions at the singular point can be represented in two forms: (i) by an equation containing the spectral parameter, or (ii) by an equation with second spatial derivative (or the first time derivative). As far as we know, such a couple of spectral problems are found for the first time.

Thus the spectral analysis of the model under consideration is accomplished in full.

Without question, the spectral analysis of the same model describing the plasma layer of other forms, for example, circular infinite cylinder or sphere, is also of interest.

Acknowledgments

One of the authors, VVN, is indebted to Professor G Barton for valuable communications concerning the topics considered in this paper. This research has been supported in part by the Heisenberg–Landau Programme and by the Russian Foundation for Basic Research (grant no 03-01-00025).

References

[1] Billups E and Ciufolini M A (ed) 1993 *Buckminsterfullerenes* (New York: VCH)
[2] Dresselhaus M S, Dresselhaus G and Eklund P C 1996 *Science of Fullerenes and Carbon Nanotubes* (San Diego, CA: Academic)
[3] Kadish K M and Ruoff R (ed) 2000 *Fullerenes: Chemistry, Physics, and Technology* (New York: Wiley-Interscience)
[4] Barton G 2004 *J. Phys. A: Math. Gen.* **37** 1011
[5] Barton G 2005 *J. Phys. A: Math. Gen.* **38** 2997
[6] Fetter A L 1973 *Ann. Phys., NY* **81** 367
[7] Stratton J A 1941 *Electromagnetic Theory* (New York: McGraw-Hill)
[8] Nesterenko V V, Lambiase G and Scarpetta G 2004 *Riv. Nuovo Cimento* **27** No. 6 1–74
[9] Bordag M, Mohideen U and Mostepanenko V M 2001 *Phys. Rep.* **351** 1
[10] Vassilevich D V 2003 *Phys. Rep.* **388** 279
[11] Elizalde E, Odintsov S D, Romeo A, Bytsenko A A and Zerbini S 1994 *Zeta Regularization: Technique with Applications* (Singapore: World Scientific)
[12] Elizalde E 1995 *Ten Physical Applications of Spectral Zeta Functions* (Berlin: Springer)
[13] Kirsten K 2001 *Spectral Functions in Mathematics and Physics* (London/Boca Raton, FL: Chapman and Hall/CRC Press)
[14] Borgnis E and Papas C H 1958 Electromagnetic waveguides and Resonators *Encyclopedia of Physics* vol 16 ed S Flüge (Berlin: Springer) pp 285–422
[15] Phillips M 1962 Classical electrodynamics *Encyclopedia of Physics* vol 4 ed S Flüge (Berlin: Springer) pp 1–108
[16] Frenkel J I 1935 *Electrodynamics* vol 2 (Moscow: ONTI) (in Russian)
[17] Barton G 1979 *Rep. Prog. Phys.* **42** 963
[18] Ronveaux A 1982 Mathematical properties of surface plasmons and van der Waals interactions in some geometries *Electromagnetic Surface Modes* ed A D Boardman (New York: Wiley)
[19] Van Kampen N G, Nijboer B R A and Schram K 1968 *Phys. Lett.* A **26** 307
[20] Gerlach E 1971 *Phys. Rev.* B **4** 393
[21] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden M 1988 *Solvable Models in Quantum Mechanics* (New York: Springer)
[22] Barton G 1985 *J. Phys. A: Math. Gen.* **18** 479
[23] Newton R G 1983 *J. Math. Phys.* **24** 2152
Newton R G 1984 *J. Math. Phys.* **25** 2991
[24] Gradshteyn I S and Ryzhik I M 2000 *Table of Integrals, Series, and Products* 6th edn (New York: Academic)
[25] Abramowitz M and Stegun I 1972 *Handbook of Mathematical Functions* (New York: Dover)
[26] Stewartson K and Waechter R T 1971 *Proc. Camb. Phil. Soc.* **69** 353
[27] Nesterenko V V, Pirozhenko I G and Dittrich J 2003 *Class. Quantum Grav.* **20** 431
[28] Pirozhenko I G, Nesterenko V V and Bordag M 2005 *J. Math. Phys.* **46** 042305
[29] Smirnov V I 1964 *A Course of Higher Mathematics* vol 4 (New York: Pergamon)
[30] Smirnov V I 1964 *A Course of Higher Mathematics* vol 2 (New York: Pergamon)
[31] Wittaker E T and Watson G N 1969 *A Course of Modern Analysis* (Cambrige: Cambridge University Press)
[32] Erdelyi A (ed) 1954 *Tables of Integral Transformations* vol 1 (New York: McGraw-Hill)
[33] Albeverio S, Brzeźniak Z and Dąbrowski L 1995 *J. Funct. Anal.* **130** 220
[34] Gaveau B and Schulman L S 1986 *J. Phys. A: Math. Gen.* **19** 1833
Manoukian E B 1989 *J. Phys. A: Math. Gen.* **22** 67
Scarlatti S and Teta A 1990 *J. Phys. A: Math. Gen.* **23** L1033
[35] Solodukhin S N 1999 *Nucl. Phys.* B **541** 461